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A direct approach to the construction of standard and non-standard Lagrangians for dissipative-like dynamical systems with variable coefficients

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Abstract

We present a direct approach to the construction of Lagrangians for a large class of one-dimensional dynamical systems with a simple dependence (monomial or polynomial) on the velocity. We rederive and generalize some recent results and find Lagrangian formulations which seem to be new. Some of the considered systems (e.g. motions with the friction proportional to the velocity and to the square of the velocity) admit infinite families of different explicit Lagrangian formulations.

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1. Introduction

In recent papers [1, 2] the problem of finding the Lagrangian description for a large class of one-dimensional dissipative (or dissipative-looking) systems was discussed. The discussion was far from being exhaustive. In this paper we present a different, more direct, approach to the problem of the construction of Lagrangians for dissipative (or dissipative-looking) systems. We simply assume some general form of the Lagrangian and then check the resulting Euler–Lagrange equations. As a result we get large families of equations admitting Lagrangian formulations. Most of these equations can be interpreted as damped, dissipative or, at least, dissipative-like systems.

The inverse problem of Lagrangian mechanics is concerned with the question of whether a given system of second-order ordinary differential equations $\ddot{q}^i = f^i(t, q, \dot{q})$ can be derived from a variational principle [3]. In other words, one tries to find a Lagrangian for this system. This problem was studied in 19th century by Helmholtz (see [4]) and by Darboux who proved that in the one-dimensional case the Lagrangian always exists [5]. The inverse problem in the two-dimensional case was solved by Douglas [6], while the general case has been completed recently [7], see also [8].

In the one-dimensional case the Lagrangian description is highly non-unique (although it is not easy to obtain corresponding Lagrangians explicitly). The problem reduces to finding the so-called Jacobi last multiplier by solving an appropriate partial differential equation of the first order. These classical results have been reconsidered recently by Nucci and Leach [9–11].

Dissipative systems were long believed to be 'beyond variational treatment' [12], which is to some extent true if we insist on the physical interpretation of the Hamiltonian and canonical momenta, compare [13]. However, by relaxing these requirements one can obtain the variational interpretation of numerous dissipative system [14–17].

In this paper we focus on more elementary issues, namely on providing explicit Lagrangian description for a large class of one-dimensional differential equations of second order with a simple (e.g. polynomial) dependence on the velocity $\dot{x}$.

2. Standard Lagrangians

Standard Lagrangians (known also as 'natural' or 'of mechanical type') are quadratic forms with respect to $\dot{x}$ (the dot denotes the differentiation with respect to t). In the one-dimensional case we can easily obtain all equations of motions corresponding to standard Lagrangians. We assume

$$\mathcal{L} = \tfrac{1}{2}P(x,t)\dot{x}^2 + Q(x,t)\dot{x} + R(x,t). \tag{1}$$

The Euler–Lagrange equations yield

$$\ddot{x} + \frac{P_x}{2P}\dot{x}^2 + \frac{P_t}{P}\dot{x} + \frac{Q_t - R_x}{P} = 0, \tag{2}$$

where subscripts x, t denote partial derivatives. As a consequence we immediately obtain the following proposition.

Proposition 2.1. *The equation of motion*

$$\ddot{x} + a(x,t)\dot{x}^2 + b(x,t)\dot{x} + c(x,t) = 0 \tag{3}$$

admits a Lagrangian description with a standard Lagrangian (1) iff

$$b_x = 2a_t. \tag{4}$$

Then $P = \exp\left(2\int^x a(\xi,t)\,\mathrm{d}\xi\right)$ *and*

$$R = \int^x (Q_t(\xi,t) - c(\xi,t)P(\xi,t))\,\mathrm{d}\xi, \tag{5}$$

where $Q = Q(x,t)$ *is an arbitrary function.*

We remark that exactly the same class of equations was studied by Euler and Jacobi (see [18] and references quoted therein).

Corollary 2.2. *Special cases of proposition 2.1:*

(1) $P = P(t)$ *and* $Q \equiv 0$*:*

$$\ddot{x} + b(t)\dot{x} + c(x,t) = 0 \quad \Longrightarrow \quad \mathcal{L} = \left(\frac{1}{2}\dot{x}^2 - \int^x c(\xi,t)\,\mathrm{d}\xi\right)\mathrm{e}^{\int^t b(\tau)\,\mathrm{d}\tau}.$$

This is a generalization of proposition 1 from [2]. In the case of linear equations (i.e. $c = x\tilde{c}(t)$*) we have*

$$\mathcal{L} = \left(\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\tilde{c}(t)x^2\right)\mathrm{e}^{\int^t b(\tau)\,\mathrm{d}\tau}. \tag{6}$$

In particular, we rederive the well-known result [19, 20] for the damped harmonic oscillator:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \quad\Longrightarrow\quad \mathcal{L} = \tfrac{1}{2}\,\mathrm{e}^{\gamma t}\big(\dot{x}^2 - \omega_0^2 x^2\big).$$

(2) $P = P(x)$ and $R \equiv 0$:

$$\ddot{x} + a(x)\dot{x}^2 + c(x,t) = 0 \quad\Longrightarrow\quad \mathcal{L} = \left(\frac{1}{2}\dot{x}^2 + \dot{x}\int^t c(x,\tau)\,\mathrm{d}\tau\right)\mathrm{e}^{2\int^x a(\xi)\,\mathrm{d}\xi}.$$

This formula simplifies for $c = c(x)$ (the case considered in [1, 2]). Then

$$\mathcal{L} = \big(\tfrac{1}{2}\dot{x}^2 + t\dot{x}c(x)\big)\,\mathrm{e}^{2\int^x a(\xi)\,\mathrm{d}\xi}. \tag{7}$$

(3) $P = P(x)$ and $Q \equiv 0$:

$$\ddot{x} + a(x)\dot{x}^2 + c(x,t) = 0 \quad\Longrightarrow\quad \mathcal{L} = \frac{1}{2}\dot{x}^2\,\mathrm{e}^{2\int^x a(\xi)\,\mathrm{d}\xi} - \int^x c(\xi,t)\,\mathrm{e}^{2\int^\xi a(y)\,\mathrm{d}y}\,\mathrm{d}\xi.$$

This is a generalization of the main result of [1] and proposition 3 from [2], where $c = c(x)$. Thus, these results are extended on the t-dependent function $c = c(x,t)$.

(4) $P = A(x)B(t)$:

$$\ddot{x} + a(x)\dot{x}^2 + b(t)\dot{x} + c(x,t) = 0 \quad\Longrightarrow\quad \mathcal{L} \textit{ is given by (1), where}$$

$P = AB$, $A = \exp(2\int^x a(\xi)\,\mathrm{d}\xi)$, $B = \exp(\int^t b(\tau)\,\mathrm{d}\tau)$, R is given by (5) and Q is arbitrary.

Example 2.3 (A particle accreting mass in a potential field). We proceed to physical aspects of the equation

$$\ddot{x} + b(t)\dot{x} + c(x,t) = 0. \tag{8}$$

Following [21], where the damped harmonic oscillator is interpreted as harmonic oscillator with time-dependent mass, we define

$$m(t) = \mathrm{e}^{\int^t b(\tau)\,\mathrm{d}\tau}, \qquad \text{i.e.} \quad b(t) = \frac{\dot{m}}{m}. \tag{9}$$

Then,

$$\mathcal{L} = \frac{1}{2}m(t)\dot{x}^2 - m(t)V(x,t), \qquad H = \frac{p^2}{2m(t)} + m(t)V(x,t), \tag{10}$$

where $V(x,t) = \int^x c(\xi,t)\,\mathrm{d}\xi$. Therefore, equation (8) can be considered either as a dissipative system or a particle with a prescribed mass time dependence in an arbitrary potential (possibly time dependent).

The next two physical examples were presented in [2]. We show that our direct approach works also in these cases. The obtained Lagrangians have a simpler form than Lagrangians found in [2].

Example 2.4 (Pendulum with increasing length). The equation of motion for the simple (nonlinear) pendulum with linearly increasing length is given by (compare [2])

$$\ddot{\theta} + \frac{2a\dot{\theta}}{l_0 + at} + \frac{g\sin\theta}{l_0 + at} = 0, \tag{11}$$

where l_0, a and g are constant. Using corollary 2.2 (case (1)) we get

$$\mathcal{L} = \tfrac{1}{2}(l_0 + at)^2\dot{\theta}^2 + (l_0 + at)g\cos\theta. \tag{12}$$

This Lagrangian is very natural and has a straightforward physical motivation (the constant length is replaced by a variable length). The increasing length effectively acts as a damping, see the second term in (11). We point out that the Lagrangian obtained by Musielak (see [2], formula (20)) is more complicated, although equivalent to (12).

Example 2.5 (Lane–Emden equation). A self-graviting spherically symmetric object (a star) composed of a fluid with the polytropic index κ is described by the Lane–Emden equation [22]:

$$\frac{d^2\psi}{d\xi^2} + \frac{2}{\xi}\frac{d\psi}{d\xi} + \psi^\kappa = 0, \tag{13}$$

where ξ, ψ are related, respectively, to the radius and density of the star. Using corollary 2.2 (case (1)) we obtain

$$\mathcal{L} = \frac{1}{2}\xi^2\left(\frac{d\psi}{d\xi}\right)^2 - \frac{\psi^{\kappa+1}\xi^2}{\kappa+1}, \tag{14}$$

compare [2], formula (22), where an equivalent (but more complicated) form of $\mathcal{L}$ was found.

All systems described by the standard Lagrangian (1) have also the Hamiltonian description. Indeed, computing generalized momentum

$$p = P\dot{x} + Q, \qquad \dot{x} = \frac{p-Q}{P}, \tag{15}$$

we easily get the standard Hamiltonian $H = p\dot{x} - \mathcal{L}$:

$$H(x,p,t) = \frac{(p-Q(x,t))^2}{2P(x,t)} - R(x,t). \tag{16}$$

In general, these Hamiltonians are t dependent, i.e. they are not the integrals of motion. The integrable cases are obtained for P, Q and R depending only on x (and in such case we can assume, without loss of the generality, $Q = 0$).

Corollary 2.6. *The equation $\ddot{x} + a(x)\dot{x}^2 + c(x) = 0$ has the standard Lagrangian formulation for any a, c. The corresponding Hamiltonian,*

$$H = \frac{1}{2}p^2 \exp\left(-2\int^x a(\xi)\,d\xi\right) + \int^x c(\xi)\exp\left(2\int^\xi a(y)\,dy\right)d\xi, \tag{17}$$

is an integral of motion.

3. Reciprocal Lagrangians

Reciprocal Lagrangians (i.e. inverses of standard-like Lagrangians) were introduced and studied recently [2, 23, 24]. If

$$\mathcal{L} = \frac{1}{L}, \qquad L = L(x,\dot{x},t)), \tag{18}$$

then

$$\ddot{x} = \frac{2\dot{x}\frac{\partial L}{\partial \dot{x}}\frac{\partial L}{\partial x} - \dot{x}L\frac{\partial^2 L}{\partial \dot{x}\partial x} + 2\frac{\partial L}{\partial t}\frac{\partial L}{\partial \dot{x}} - L\frac{\partial^2 L}{\partial t\partial \dot{x}} + L\frac{\partial L}{\partial x}}{L\frac{\partial^2 L}{\partial \dot{x}^2} - 2\left(\frac{\partial L}{\partial \dot{x}}\right)^2}. \tag{19}$$

We confine ourselves to $\mathcal{L}$ of the form

$$\mathcal{L} = \frac{1}{L}, \qquad L = F(x,t)\dot{x}^{\nu} + G(x,t). \tag{20}$$

Substituting (20) into (19) we obtain

$$\ddot{x} = \frac{p\dot{x}^{2\nu} + q\dot{x}^{2\nu-1} + r\dot{x}^{\nu} + s\dot{x}^{\nu-1} + w}{g\dot{x}^{\nu-2} - h\dot{x}^{2\nu-2}}, \tag{21}$$

where

$$p := (1+\nu)FF_x, \qquad q := \nu F F_t, \qquad r := (1+2\nu)FG_x + (1-\nu)F_x G,$$
$$s := 2\nu G_t F - \nu G F_t, \qquad w := GG_x, \qquad g := \nu(\nu-1)FG, \qquad h := \nu(\nu+1)F^2.$$

3.1. Linear case

We first consider the case linear in $\dot{x}$, i.e. $\nu = 1$. The case $\nu = 1$, $F = 1$ is considered in [23], with a special stress on G quadratic in x (leading to second-order Riccati equations), see also [25] for a general discussion of this case.

In the case $\nu = 1$ equation (21) reduces to a special case of (3):

$$\ddot{x} = -\frac{F_x}{F}\dot{x}^2 - \frac{(F_t + 3G_x)}{2F}\dot{x} - \frac{(2G_tF - GF_t + GG_x)}{2F^2}. \tag{22}$$

First, we confine ourselves to t-independent F and G. Then, the coefficients a, b and c by powers of $\dot{x}$ depends on x only. They are not independent. Indeed,

$$a = \frac{F'}{F}, \qquad b = \frac{3G'}{2F}, \qquad c = \frac{GG'}{2F^2}, \tag{23}$$

where the prime denotes the differentiation with respect to x. Hence, substituting $G = 3cF/b$ and $F' = aF$ to the last equation of (23), we get a constraint on a, b and c, see (25).

Proposition 3.1. *The equation*

$$\ddot{x} + a(x)\dot{x}^2 + b(x)\dot{x} + c(x) = 0 \tag{24}$$

admits a Lagrangian description with $\mathcal{L} = (\dot{x}F(x) + G(x))^{-1}$ *iff*

$$c_{,x} + \left(a - \frac{b_{,x}}{b}\right)c = \frac{2}{9}b^2. \tag{25}$$

Then, $F(x) = \exp(\int^x a(\xi)\,\mathrm{d}\xi)$ *and* $G(x) = 3c(x)F(x)/b(x)$.

Therefore, we can choose arbitrary functions $a(x)$, $b(x)$ and then c have to satisfy equation (25). Solving this equation we get

$$c(x) = \frac{2}{9}b(x)\int^x b(\xi)\exp\left(\int_x^{\xi} a(y)\,\mathrm{d}y\right)\mathrm{d}\xi. \tag{26}$$

Another (more general) Hamiltonian formulation for equation (24) was found in [27] by the Prelle–Singer method, compare also [39]. The case $a = 0$ corresponds to a class of modified Emden-type equations.

Example 3.2 (Liénard-type nonlinear oscillator). Taking $a = 0$ and $b(x) = kx$ ($k = \text{const}$) we obtain

$$c(x) = \frac{2}{9}kx\left(\frac{1}{2}kx^2 + \lambda\right) = \frac{k^2x^3}{9} + \lambda_1 x, \tag{27}$$

where $\lambda = \text{const}$ and $\lambda_1 := \frac{2}{9}k\lambda$. This case corresponds exactly to a Liénard-type nonlinear oscillator which shows very unusual properties, like isochronous oscillations for $\lambda_1 > 0$ [24]. In this case the Lagrangian is given by

$$\mathcal{L} = \frac{1}{\dot{x} + \frac{1}{3}kx^2 + \frac{2}{3}\lambda}, \tag{28}$$

compare [2, 24].

Another possibility is to choose the arbitrary functions $b(x)$, $c(x)$ and then $a(x)$ is given by

$$a = \frac{b_{,x}}{b} - \frac{c_{,x}}{c} + \frac{2b^2}{9c}. \tag{29}$$

Proposition 3.1 generalizes propositions 4 and 5 from [2] (and coincides with proposition 2 from [25]).

The case $\nu = 1$ contains other interesting subcases.

Example 3.3. Taking $F(t) = f_0 \mathrm{e}^{2kt}$ and $G(t) = g_0 \mathrm{e}^{kt}$ we reduce (22) to $\ddot{x} + k\dot{x} = 0$.

A next case is obtained by the assumption $F = f(t)$, $G = xg(t)$. Then, equation (22) reduces to the linear equation:

$$\ddot{x} + b(t)\dot{x} + c(t)x = 0, \tag{30}$$

where

$$b = \frac{\dot{f} + 3g}{2f}, \qquad c = \frac{2f\dot{g} - g\dot{f} + g^2}{2f^2}. \tag{31}$$

The system (31) expresses b, c in terms of f, g. These equations cannot be inverted explicitly (we correct here a mistake made in our preprint [26]). Given b, c we may try to compute corresponding f, g. Substituting $g = \frac{2}{3}fb - \frac{1}{3}\dot{f}$ into the second equation we obtain a Riccati equation, see [25] (proposition 1). Therefore, we have a reciprocal Lagrangian for (30) but in an implicit form (the Lagrangian is expressed in terms of a solution of the Riccati equation) [25]. In section 3.3 we extend this result obtaining a one-parameter family of non-standard Lagrangians.

3.2. Quadratic case

In the case $\nu = 2$ the Lagrangian (20) yields more complicated equation:

$$\ddot{x} = \frac{3FF_x\dot{x}^4 + 2FF_t\dot{x}^3 + (5FG_x - F_xG)\dot{x}^2 + (4G_tF - 2GF_t)\dot{x} + GG_x}{2F(G - 3F\dot{x}^2)}. \tag{32}$$

In the particular case $G = 0$ we get

$$\ddot{x} + \frac{F_x}{2F}\dot{x}^2 + \frac{F_t}{3F}\dot{x} = 0, \tag{33}$$

which yields the following proposition.

Proposition 3.4. *The equation*

$$\ddot{x} + a(x,t)\dot{x}^2 + b(x,t)\dot{x} = 0 \tag{34}$$

admits a Lagrangian description with the Lagrangian proportional to $\dot{x}^{-2}$ *iff* $3b_x = 2a_t$. *Then,* $\mathcal{L} = \left(\dot{x} \exp \int^x a(\xi, t)\, \mathrm{d}\xi\right)^{-2}$.

Another interesting particular case is given by $G = G(t)$, $F = f(x)G^3$. We obtain

$$\ddot{x} = -\frac{f'}{2f}\dot{x}^2 - \frac{\dot{G}}{G}\dot{x}, \tag{35}$$

which is a particular case of (34).

Proposition 3.5. *The equation*

$$\ddot{x} + a(x)\dot{x}^2 + b(t)\dot{x} = 0 \tag{36}$$

admits a Lagrangian description with a Lagrangian given by $\mathcal{L} = (F\dot{x}^2 + G)^{-1}$, *where* $G(t) = \exp\left(\int^t b(\tau)\,\mathrm{d}\tau\right)$, $F(x,t) = \exp\left(3\int^t b(\tau)\,\mathrm{d}\tau + 2\int^x a(\xi)\,\mathrm{d}\xi\right)$.

Therefore, equation (36) admits at least three different explicit Lagrangian descriptions: standard one (compare case 4 of corollary 2.2) and two reciprocal.

Example 3.6 (Buchdahl equation). A particular example of class (36) is the Buchdahl equation of the general relativity, compare [27, 28],

$$\ddot{x} = \frac{3\dot{x}^2}{x} + \frac{\dot{x}}{t}, \tag{37}$$

which corresponds to $a = -3/x$, $b = -1/t$. Then, propositions 3.4 and 3.5 give the following Lagrangians:

$$\mathcal{L}_1 = k_1\dot{x}^{-2}t^3x^6, \qquad \mathcal{L}_2 = \frac{1}{k_1(k_2\dot{x}^2x^6t^3 + t)}, \tag{38}$$

where k_1, k_2 are constant. We remark that (37) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{x^2}\right)\right) = 0,$$

which yields the general solution in the form $c_1x^2t^2 + c_2x^2 = 1$.

3.3. A generalization of the reciprocal case

After completing the first version of this work [26] we realized that practically all results of section 3.1 have been obtained earlier by Musielak [25]. In this section we improve and generalize some of these results. We consider the following generalization of reciprocal Lagrangians:

$$\mathcal{L} = \frac{1}{L^m}, \qquad L = L(x, \dot{x}, t)), \tag{39}$$

where m is a real constant (compare [29] where this ansatz is applied in a particular case). Then Euler–Lagrange equations read

$$\ddot{x} = \frac{(m+1)\left(\dot{x}\frac{\partial L}{\partial \dot{x}}\frac{\partial L}{\partial x} + \frac{\partial L}{\partial t}\frac{\partial L}{\partial \dot{x}}\right) - \dot{x}L\frac{\partial^2 L}{\partial \dot{x}\partial x} - L\frac{\partial^2 L}{\partial t\partial \dot{x}} + L\frac{\partial L}{\partial x}}{L\frac{\partial^2 L}{\partial \dot{x}^2} - (m+1)\left(\frac{\partial L}{\partial \dot{x}}\right)^2}. \tag{40}$$

We confine ourselves to L linear in $\dot{x}$ considering two interesting subcases. First, we assume t-independent L, i.e.

$$\mathcal{L} = \frac{1}{L^m}, \qquad L = F(x)\dot{x} + G(x). \tag{41}$$

The Euler–Lagrange equations read

$$\ddot{x} + \frac{F'}{F}\dot{x}^2 + \frac{(m+2)G'}{(m+1)F}\dot{x} + \frac{GG'}{(m+1)F^2} = 0, \tag{42}$$

and we easily obtain an analogue of proposition 3.1.

Proposition 3.7. *The equation* $\ddot{x} + a(x)\dot{x}^2 + b(x)\dot{x} + c(x) = 0$ *admits the Lagrangian description with* $\mathcal{L} = (\dot{x}F(x) + G(x))^{-m}$ *iff*

$$c_{,x} + \left(a - \frac{b_{,x}}{b}\right)c = \frac{(m+1)}{(m+2)^2}b^2. \tag{43}$$

Then, $F(x) = \exp\left(\int^x a(\xi)\,\mathrm{d}\xi\right)$ *and* $G(x) = (m+2)c(x)F(x)/b(x)$.

In order to obtain another proposition, we consider L linear both in $\dot{x}$ and x,

$$\mathcal{L} = \frac{1}{L^m}, \qquad L = f(t)\dot{x} + g(t)x. \tag{44}$$

Substituting (44) into (40) we obtain

$$\ddot{x} + \frac{m\dot{f} + (m+2)g}{(m+1)f}\dot{x} + \frac{(m+1)f\dot{g} - g\dot{f} + g^2}{(m+1)f^2}x = 0. \tag{45}$$

We identify this equation with (30),

$$b = \frac{m\dot{f} + (m+2)g}{(m+1)f}, \qquad c = \frac{(m+1)f\dot{g} - g\dot{f} + g^2}{(m+1)f^2}. \tag{46}$$

Similarly as at the end of section 3.1, we try to express f, g in terms of b, c. Now, we have at our disposal the free parameter m. From the first equation we compute

$$g = \frac{(m+1)fb - m\dot{f}}{m+2}, \tag{47}$$

and substituting it into the second equation we get

$$c = \frac{(m+1)(ub + \dot{b}) - m(\dot{u} + u^2)}{m+2} + \frac{mu^2 - (m+1)ub}{(m+1)(m+2)} + \frac{((m+1)b - mu)^2}{(m+1)(m+2)^2},$$

where u is defined by $\dot{f} = uf$. Thus, we obtained a Riccati equation for u:

$$\dot{u} + \frac{mu^2}{m+2} - \frac{mbu}{m+2} - \frac{(m+1)b^2}{m(m+2)} - \frac{(m+1)b}{m} + \frac{(m+2)c}{m} = 0.$$

In the special case $m = 1$ this equation coincides with equation (8) of [25].

Proposition 3.8. *Equation (30) (for any* $b(t), c(t)$*) admits an explicit Lagrangian description with the generalized reciprocal Lagrangian of the form* $\mathcal{L} = (\dot{x}f(t) + xg(t))^{-m}$*. Functions* f, g *can be expressed in terms of* b, c *by a Riccati equation.*

Therefore, any equation of the form $\ddot{x} + b(t)\dot{x} + c(t)x = 0$ (including equations of mathematical physics, like Airy, Bessel, Hermite or Legendre equation) has an explicit standard Lagrangian (see (6)) and one-parameter family of implicit generalized reciprocal Langrangians. Actually, there are two generalized reciprocal Lagrangians corresponding to two values of the parameter m.

4. Lagrangians with a modified kinetic term

In this section we consider generalizations of standard Lagrangians, where the kinetic term $\dot{x}^2$ is replaced by some more general expression (and the term linear in $\dot{x}$ is absent).

4.1. Monomial case

First, we assume the monomial case:

$$\mathcal{L} = F(x,t)\dot{x}^{\mu} - G(x,t). \tag{48}$$

The equation of motion reads

$$\ddot{x} = -\frac{\dot{x}^2 F,_x}{\mu F} - \frac{\dot{x} F,_t}{(\mu-1)F} - \frac{\dot{x}^{2-\mu} G,_x}{\mu(\mu-1)F}. \tag{49}$$

Proposition 4.1. *The equation of motion*

$$\ddot{x} + a(x,t)\dot{x}^2 + b(x,t)\dot{x} + c(x,t)\dot{x}^{2-\mu} = 0 \qquad (\mu \neq 0, 1) \tag{50}$$

admits a Lagrangian description with the Lagrangian (48) iff

$$(\mu-1)b,_x = \mu a,_t\,. \tag{51}$$

Then, $F = \exp\left(\mu \int^x a(\xi,t)\,\mathrm{d}\xi\right)$ *and* $G = \mu(\mu-1)\int^x c(\xi,t)F(\xi,t)\,\mathrm{d}\xi$.

The proof follows directly by comparing (50) with (49). Another result is obtained by assuming $F = F(x)$ and $G = G(x)$.

Proposition 4.2. *The equation*

$$\ddot{x} = -a(x)\dot{x}^2 - c(x)\dot{x}^{\nu} \qquad (\nu \neq 1, 2) \tag{52}$$

admits for any $a(x), c(x)$ *a Lagrangian description. The Lagrangian reads*

$$\mathcal{L} = F(x)\dot{x}^{2-\nu} - G(x), \tag{53}$$

where

$$F(x) = \exp\left((2-\nu)\int^x a(\xi)\,\mathrm{d}\xi\right), \qquad G(x) = (2-\nu)(1-\nu)\int^x c(\xi)F(\xi)\,\mathrm{d}\xi.$$

Corollary 4.3. *Taking* $c(x) = 0$, $a(x) = k = \mathrm{const}$ *and denoting* $n = \nu - 2$, *we obtain (for* $n \neq 0$*)*

$$\ddot{x} + k\dot{x}^2 = 0 \quad \Longrightarrow \quad \mathcal{L} = C\dot{x}^n\,\mathrm{e}^{nkx}.$$

4.2. General case

Let us consider a class of standard-like Lagrangians with quadratic kinetic terms replaced by an arbitrary smooth function of $\dot{x}$:

$$\mathcal{L} = F(x,t)\psi(\dot{x}) + G(x,t). \tag{54}$$

The equation of motion reads

$$\ddot{x} + \frac{(F_t + \dot{x}F_x)\psi' - F_x\psi - G_x}{F\psi''} = 0. \tag{55}$$

Assuming $F = F_0 = \text{const}$, we obtain the equation

$$\ddot{x} = \frac{G_x}{F_0 \psi''}, \tag{56}$$

where the right-hand side is of the form $f(x,t)\phi(\dot{x})$ for some functions f, ϕ. Indeed, it is enough to take $G_x = f F_0$ oraz $\psi'' = 1/\phi$.

Proposition 4.4. *The equation $\ddot{x} = f(x,t)R(\dot{x})$ admits a Lagrangian description with the Lagrangian $L = \Psi(\dot{x}) + G(x,t)$, where*

$$\Psi(v) := \int^{v} \mathrm{d}\eta \int^{\eta} \frac{\mathrm{d}\xi}{R(\xi)}, \quad G(x,t) = \int^{x} f(\xi,t)\,\mathrm{d}\xi, \tag{57}$$

(provided that the above integrals exist).

Corollary 4.5. *Special cases of proposition 4.4:*

(1) $\ddot{x} = \dot{x} f(x,t) \quad \Longrightarrow \quad \mathcal{L} = \dot{x}\ln|\dot{x}| + \int^{x} f(\xi,t)\,\mathrm{d}\xi.$

(2) $\ddot{x} = \dot{x}^2 f(x,t) \quad \Longrightarrow \quad \mathcal{L} = -\ln|\dot{x}| + \int^{x} f(\xi,t)\,\mathrm{d}\xi.$

(3) $\ddot{x} = -k_0\dot{x}^{\nu} \quad \Longrightarrow \quad \mathcal{L} = \dfrac{\dot{x}^{2-\nu}}{(2-\nu)(1-\nu)} - k_0 x$ *(*$\nu \neq 1, 2$*).*

(4) $\ddot{x} = f(x,t)\left(1 - \dfrac{\dot{x}^2}{c^2}\right)^{3/2} \quad \Longrightarrow \quad \mathcal{L} = -c^2\sqrt{1 - \dfrac{\dot{x}^2}{c^2}} + \int^{x} f(\xi,t)\,\mathrm{d}\xi.$

The last case of corollary 4.5 describes a relativistic particle in a prescribed potential field. Indeed, the equation of motion can be obviously rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{x}}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} = f(x,t). \tag{58}$$

Example 4.6 (Relativistic particle in a dissipative medium). Applying proposition 4.4 to the case defined by $f(x,t) = 1$ and $R(\dot{x})$ given by

$$R(\dot{x}) = g(\dot{x})\left(1 - \frac{\dot{x}^2}{c^2}\right)^{3/2}, \tag{59}$$

where $g(\dot{x})$ is a prescribed function, we get the equation of motion for a relativistic particle with a dissipation or/and damping:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{x}}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} = g(\dot{x}). \tag{60}$$

The corresponding Lagrangian can always be found in quadratures (but explicit formulation is usually unknown or non-existing). A special case of equation (60) was considered in [36].

5. Radical Lagrangians

We consider Lagrangians of the form

$$\mathcal{L} = \sqrt[\mu]{A(x,t)\dot{x}^\nu + B(x,t)}. \tag{61}$$

The Euler–Lagrange equations yield

$$\ddot{x} = \frac{p\dot{x}^{2\nu} + q\dot{x}^{2\nu-1} + r\dot{x}^\nu + s\dot{x}^{\nu-1} + w}{g\dot{x}^{2\nu-2} + h\dot{x}^{\nu-2}}, \tag{62}$$

where

$$\begin{aligned} g &:= \frac{(\nu-\mu)}{(1-\mu)}A, \qquad h := \frac{\mu(\nu-1)}{(1-\mu)}B, \\ p &:= -\frac{(\nu+\mu)}{\nu(1-\mu)}A_x, \qquad q := -\frac{1}{(1-\mu)}A_t, \\ r &:= -\frac{(\nu-\nu\mu-\mu)}{\nu(1-\mu)}B_x - \frac{\mu(\nu+1)}{\nu(1-\mu)}\frac{A_x B}{A}, \\ s &:= -B_t - \frac{\mu}{(1-\mu)}\frac{A_t B}{A}, \qquad w := \frac{\mu}{\nu(1-\mu)}\frac{B_x B}{A}. \end{aligned} \tag{63}$$

In this paper we will assume either $\mu = \nu \neq 1$ or $\mu \neq \nu = 1$. In those cases the denominator simplifies and the right-hand side of (62) is a polynomial in $\dot{x}$.

5.1. *The case* $\mu = \nu \neq 1$

In this case equation (62) reduces to

$$\ddot{x} = \frac{\frac{2A_x}{B}\dot{x}^{\nu+2} + \frac{A_t}{B}\dot{x}^{\nu+1} + \left(\frac{(1+\nu)A_x}{A} - \frac{\nu B_x}{B}\right)\dot{x}^2 + \left(\frac{(1-\nu)B_t}{B} + \frac{\nu A_t}{A}\right)\dot{x} + \frac{B_x}{A}\dot{x}^{2-\nu}}{\nu(1-\nu)}. \tag{64}$$

A further reduction is obtained by assuming that $A = A(t)$, $B = B(t)$. Then, equation (64) becomes

$$\ddot{x} = -\left(\frac{\dot{A}}{(\nu-1)A} - \frac{\dot{B}}{\nu B}\right)\dot{x} - \frac{\dot{A}}{\nu(\nu-1)B}\dot{x}^{\nu+1}. \tag{65}$$

Proposition 5.1. *The equation*

$$\ddot{x} = -a(t)\dot{x} - b(t)\dot{x}^{\nu+1} \tag{66}$$

admits (for $\nu \neq 0$, $\nu \neq 1$ *and any functions* a, b*) a Lagrangian description with the Lagrangian of the form* $\mathcal{L} = \sqrt[\nu]{A(t)\dot{x}^\nu + B(t)}$*, where*

$$\begin{aligned} A(t) &= \left(\nu\int^t b(\tau)\exp\left(-\nu\int^\tau a(y)\,\mathrm{d}y\right)\mathrm{d}\tau\right)^{1-\nu}, \\ B(t) &= \left(\nu\int^t b(\tau)\exp\left(-\nu\int^\tau a(y)\,\mathrm{d}y\right)\mathrm{d}\tau\right)^{-\nu}\exp\left(-\nu\int^t a(\tau)\,\mathrm{d}\tau\right). \end{aligned} \tag{67}$$

In order to proof this proposition it is enough to compare (66) with (65) and to solve resulting differential equations.

Two interesting special cases can be obtained by requiring either $b = 0$ (i.e. $A(t) = \text{const}$) or $a = 0$ (i.e. $\nu \ln A - (\nu - 1)\ln B = \text{const}$).

Corollary 5.2. *Special cases of proposition 5.1:*

1. $A = A_0 = \text{const}$, $B = B(t)$:

$$\ddot{x} + a(t)\dot{x} = 0 \quad \Longrightarrow \quad \mathcal{L} = \sqrt[\nu]{A_0\dot{x}^\nu + B_0 \exp\left(-\nu \int^t b(\tau)\,\mathrm{d}\tau\right)},$$

where $B_0 = \text{const}$ *and* $\nu \neq 0, 1$.

2. $B = c_0 A^{\frac{\nu}{\nu-1}}$,

$$\ddot{x} + b(t)\dot{x}^m = 0 \quad \Longrightarrow \quad \mathcal{L} = \sqrt[m+1]{F^{-m}\dot{x}^{m+1} + c_0 F^{-m-1}},$$

where $m \neq 1, 2$, $F = F(t) = -c_0(m+1)\int^t b(\tau)\,\mathrm{d}\tau$ *and* $c_0 = \text{const}$.

5.2. *The case* $\mu \neq \nu = 1$

In this case equation (62) reduces to

$$\ddot{x} = \frac{A_x(1+\mu)\dot{x}^2 + \left(A_t + B_x(1-2\mu) + \frac{2\mu A_x B}{A}\right)\dot{x} + B_t(1-\mu) + \frac{\mu A_t B}{A} - \frac{\mu B_x B}{A}}{A(\mu-1)}. \tag{68}$$

We assume $A = A(t)$, $B = B(t)$. Then, equation (68) becomes

$$\ddot{x} = \frac{\dot{A}}{(\mu-1)A}\dot{x} + \frac{\mu\dot{A}B}{(\mu-1)A^2} - \frac{\dot{B}}{A}, \tag{69}$$

and, solving linear differential equations (similarly as in the case of proposition 5.1), we get the following result.

Proposition 5.3. *The equation*

$$\ddot{x} = a(t)\dot{x} + b(t) \tag{70}$$

admits (for any functions a, b*) a Lagrangian description with the Lagrangian of the form* $\mathcal{L} = \sqrt[\mu]{A(t)\dot{x} + B(t)}$ *where* $\mu \neq 1$ *and*

$$\begin{aligned} A(t) &= \exp\left((\mu-1)\int^t a(\tau)\,\mathrm{d}\tau\right), \\ B(t) &= -\left(\int^t b(\tau)\mathrm{e}^{-\int^\tau a(y)\,\mathrm{d}y}\,\mathrm{d}\tau\right)\exp\left(\mu\int^t a(\tau)\,\mathrm{d}\tau\right). \end{aligned} \tag{71}$$

We point out that for $b = 0$ formulae (71) yield $B(t) = \exp\left(\mu\int^t a(\tau)\,\mathrm{d}\tau\right)$ (the integration constant has to be taken into account).

6. Multi-Lagrangian cases

The Lagrangian of proposition 5.3 can be rewritten as

$$\mathcal{L} = \mathrm{e}^{\int^t a(\tau)\,\mathrm{d}\tau}\sqrt[\mu]{\dot{x}\mathrm{e}^{-\int^t a(\tau)\,\mathrm{d}\tau} - \int^t b(\tau)\,\mathrm{e}^{-\int^\tau a(y)\,\mathrm{d}y}}, \tag{72}$$

and this form suggests the following generalization which can be easily verified by a simple straightforward calculation.

Proposition 6.1. *The equation*

$$\ddot{x} = a(t)\dot{x} + b(t) \tag{73}$$

admits (for any functions a, b) a Lagrangian description with the Lagrangian of the form

$$\mathcal{L} = \mathrm{e}^{\int^t a(\tau)\,\mathrm{d}\tau} F\left(\dot{x}\mathrm{e}^{-\int^t a(\tau)\,\mathrm{d}\tau} - \int^t b(\tau)\,\mathrm{e}^{-\int^\tau a(y)\,\mathrm{d}y}\right), \tag{74}$$

where F is a function of one variable (such that $F'' \neq 0$).

We remark that in the case of constant a, b (damped harmonic oscillator) many other independent Lagrangians were obtained in [9].

In particular, we point out that the simple classical equation $\ddot{x} + k\dot{x} = 0$ has Lagrangians of all forms considered in our paper, namely

$$\begin{aligned} &L_1 = \frac{1}{2}\,\mathrm{e}^{kt}\dot{x}^2, \qquad L_2 = \frac{1}{\mathrm{e}^{2kt}\dot{x} + \mathrm{e}^{kt}}, \qquad L_3 = \dot{x}^\mu \mathrm{e}^{(\mu-1)kt}, \\ &L_4 = \dot{x}\ln|\dot{x}| - kx, \qquad L_5 = \sqrt[\nu]{\dot{x}^\nu + \mathrm{e}^{-\nu kt}}. \end{aligned} \tag{75}$$

However, any of these Lagrangians is equivalent (i.e. differs at most by a total derivative) to a Lagrangian of the form (74), namely

$$L_F = \mathrm{e}^{-kt} F(\dot{x}\mathrm{e}^{kt} + c_0), \tag{76}$$

with $F(\xi)$ equal to $\frac{1}{2}\xi^2$, ξ^{-1}, ξ^μ, $\xi\ln|\xi|$ and $\sqrt[\nu]{\xi}$, respectively. A one parameter family of Lagrangians for the equation $\ddot{x} + \dot{x} = 0$ was considered in [30]. All members of this family are equivalent to particular cases of (76), as well.

Another multi-Lagrangian case is described by corollary 4.3, where we present a one-parameter family of Lagrangians for the equation $\ddot{x}+k\dot{x}^2 = 0$. What is more, the corresponding Hamiltonian is proportional to the Lagrangian (for any $n \neq 0$) and is time independent. Hence, $\mathcal{L}$ is an integral of motion. This observation can be generalized as follows.

Proposition 6.2. *Suppose that a Lagrangian $\mathcal{L} = \mathcal{L}(q^i, \dot{q}^i, t)$ is an invariant of motion (i.e. $\mathrm{d}\mathcal{L}/\mathrm{d}t = 0$). Then, for any (sufficiently smooth) function $F : \mathbb{R} \to \mathbb{R}$, the Lagrangian $\tilde{\mathcal{L}} = F(\mathcal{L})$ yields the same equations of motion.*

The proof is straightforward. We compute

$$\frac{\partial\tilde{\mathcal{L}}}{\partial q^i} = \frac{\mathrm{d}F}{\mathrm{d}\mathcal{L}}\frac{\partial\mathcal{L}}{\partial q^i}, \qquad \frac{\partial\tilde{\mathcal{L}}}{\partial \dot{q}^i} = \frac{\mathrm{d}F}{\mathrm{d}\mathcal{L}}\frac{\partial\mathcal{L}}{\partial \dot{q}^i}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{\mathcal{L}}}{\partial \dot{q}^i} = \frac{\mathrm{d}^2F}{\mathrm{d}\mathcal{L}^2}\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} + \frac{\mathrm{d}F}{\mathrm{d}\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial \dot{q}^i}.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{\mathcal{L}}}{\partial \dot{q}^i} - \frac{\partial\tilde{\mathcal{L}}}{\partial q^i} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial \dot{q}^i} - \frac{\partial\mathcal{L}}{\partial q^i}\right)\frac{\mathrm{d}F}{\mathrm{d}\mathcal{L}} + \frac{\mathrm{d}^2F}{\mathrm{d}\mathcal{L}^2}\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t},$$

from which the proof follows immediately.

Taking into account proposition 6.2, we see that $\mathcal{L}_F := F(\dot{x}\mathrm{e}^{kt})$ is a Lagrangian for the equation $\ddot{x} + k\dot{x}^2 = 0$ (for any smooth function F). Another Lagrangian (time independent) for this equation was found by Sarlet: $\mathcal{L} = \dot{x}(1 - \ln\dot{x})\exp(kx)$, see [31].

7. Conclusions

In this paper we succeeded to rederive all results of [1, 2] in a straightforward, simple way. Actually, we found many other one-dimensional dissipative-looking systems possessing a Lagrangian description. One-dimensional systems admitting the Lagrangian formulation were discussed in numerous papers (see, e.g., [21, 23, 24, 31–39]); some of them devoted mostly to the damped harmonic oscillator, e.g., [9, 19, 20, 40, 41]. Surprisingly enough, using quite elementary tools, we succeeded to find some number of one-dimensional Lagrangians which seem to be overlooked in the existing literature (see, for instance, sections 3.3 and 4). Many other cases are rederived in a simpler, direct way.

Lagrangian description for some considered systems is not unique; they may possess several different, non-equivalent Lagrangians (the problem of the equivalence was discussed in [42]). This is a general property of one-dimensional systems, compare, e.g., [9]. However, the corresponding existence theorems do not provide methods for producing explicit examples.

The equations $\ddot{x} + k\dot{x} = 0$ and $\ddot{x} + k\dot{x}^2 = 0$, usually considered as classical dissipative equations (compare [13]), have infinite families of Lagrangians, see section 6. The first of these equation has Lagrangians of all forms considered in our paper, see (75).

In our paper we considered exclusively one-dimensional systems. It would be interesting to extend these results on higher dimensions.

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